TOTARO'S THEOREM ON HYPERSURFACE IRRATIONALITY

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Abstract. This note is an introduction to Totaro's theorem on hypersurface stable irrationality. We begin with an introduction to some results on rationality problems of hypersurfaces. In particular, we state the Totaro's theorem on hypersurfaces that are not stably rational, which is the main theorem we discuss in this note. We then give basic definitions and propositions needed. Then we explain detailed proof of Totaro's theorem, including the specialization method and the decomposition of the diagonal. We end with some examples and a corollary on rationality in families.

Contents

1. Introduction

The rationality problem is one of the central questions in algebraic geometry. Intuitively, it asks whether an algebraic variety X/k of dimension n can be almost parametrized by n independent coordinates, or more precisely, whether its function field is isomorphic to $k(t_1, \ldots, t_n)$. Equivalently, we say X is rational if it is birational equivalent to $\mathbb{P}^n$.

One of the simplest varieties except projective spaces $\mathbb{P}^{n+1}$ are the hypersurfaces in $\mathbb{P}^{n+1}$. These are the main objects we consider in this note. Let $X_d \subset \mathbb{P}^{n+1}$ be a smooth hypersurface of degree d and dimension n. We wonder whether X_d is rational or not. The answer of this question is rather simple when d is sufficiently large or $n = 1, 2$.

Indeed, for a smooth plane curve $C \subset \mathbb{P}^2$, we have the following genus formula:

$$g(C) = \frac{(d-1)(d-2)}{2}.$$

Combining the fact that a smooth projective curve is rational if and only if it has genus zero, we obtain eh following elementary classification:

Proposition 1. *(Curves) A smooth plane curve of degree d is rational if and only if $d = 1$ or $d = 2$.*

For surfaces in $\mathbb{P}^3$, let S_d be a smooth surface of degree d. By the adjunction formula:

$$K_{S_d} = (d-4)H|_{S_d},$$

where K_{S_d} is the canonical divisor and H is the hyperplane class.

For $d \geq 5$, the canonical divisor is effective. For $d = 4$, S_d is a $K3$ surface. In both cases, S_d cannot be rational, since a smooth rational surface has Kodaira dimension $-\infty$. On the other hand, plane and quadrics are rational, and a smooth cubic surface over and algebraic closed field is rational: it is isomorphic to the blow-up of $\mathbb{P}^2$ at 6 points in general position, see [Bir25, Page 28]. In conclusion, we have:

Theorem 1. *(Surfaces) Let $S_d \subset \mathbb{P}^3$ be a smooth surface of degree d. Then S_d is rational for $d = 1, 2, 3$, and is not rational for $d \geq 4$.*

In higher dimensional cases, the answer of rationality is far from complete. However, when the degree d is larger than $n + 2$, the Kodaira dimension, a birational invariant, obstructs these hypersurfaces to be rational. Indeed, for a degree d hypersurface $X_d \subset \mathbb{P}^{n+1}$, the adjunction formula gives

$$K_{X_d} = (d-n-2)H|_{X_d}.$$

Therefore, if $d \geq n + 2$, K_{X_d} is effective. If $d = n + 2$, then K_{X_d} is trivial, we say X_d is Calabi-Yau. If $d > n+2$, then K_{X_d} is ample, and we say X_d is of general type. In both cases, the Kodaira dimension $\kappa(X_d) \geq 0$, while a rational smooth projective variety has Kodaira dimension $-\infty$. Thus we get:

Proposition 2. *Let $X_d \subset \mathbb{P}^{n+1}$ be a smooth hypersurface of degree d. If $d \geq n+2$, then X_d is not rational.*

The difficult range is

$$d \leq n + 1.$$

In this range X_d is Fano. Campana and Kollár–Miyaoka–Mori proved independently that a smooth Fano variety is rationally connected in characteristic 0, see [Cam92] and [KMM92], that is, any two points in this variety can be connected by a rational curve:

Theorem 2. *(Campana; Kollár–Miyaoka–Mori) Let X be a smooth Fano variety over an algebraically closed field of characteristic zero. Then X is rationally connected.*

However, rational connectedness is much weaker than being rational. The question is: which Fano hypersurfaces are rational, stably rational, or irrational?

In dimension 3, the difficult part is in degree $d = 3, 4$. When $d = 3$, the problem is solved by Clemens–Griffiths [CG72] as following:

Theorem 3. *A smooth cubic threefold $X_3 \subset \mathbb{P}^4$ is not rational.*

The method of Clemens and Griffiths is Hodge theory. They associate to a smooth cubic threefold X its intermediate Jacobian. The intermediate Jacobian is an abelian variety $J(X)$ constructed as following:

$$J(X) = (H^{3,0}(X) \oplus H^{2,1}(X))^*/H^3(X;\mathbb{Z}).$$

This abelian variety has a natural polarization Θ, which is induced by the cup product on $H^3(X;\mathbb{C})$. If X is rational, then its intermediate Jacobian $J(X)$ have to be from Jacobians of curves, which is of level one. However, by comparing $J(X)$, $Alb(S)$ (the Albanese variety of family S of algebraic curves on X), and $Pic(S)$ (the Picard variety of S), they showed that $J(X)$ must of level two. So, X cannot be rational.

In degree 4, Iskovskikh–Manin proved the following theorem [IM71]:

Theorem 4. *A smooth quartic threefold $X_4 \subset \mathbb{P}^4$ is not rational.*

This requires a different method, usually understood as an early form of birational rigidity. One studies the birational self-maps and Mori fibre space structures of the quartic threefold and proves that it cannot be birational to $\mathbb{P}^n$.

In dimension 4, there is a recent striking progress showing that a very general cubic fourfold is not rational, see [KKPY25].

Theorem 5. *A very general smooth cubic fourfold is not rational.*

The method is to construct "Hodge atoms" and study how it performs under birational transform. The construction of Hodge atom highly relied on the theory of F-bundle developed by Kontsevich and his collaborators, combing noncommutative Hodge theory and mirror symmetry. The key is that the Hodge atoms change only in codimension ≥ 2 in birational transformation. However, for a very general smooth cubic fourfold X, the atom containing $H^{3,1}$ cannot come from points, curves or surfaces. While $\mathbb{P}^4$ only has one trivial atom.

Remark 1. *In the same paper, they also give an outline that when considering Hodge atom with Serre duality or with integral structure, one could give another proof of irrationality of cubic threefolds and very general quartic fourfolds.*

For higher dimensional cases, Kollar proved [Kol96] that

Theorem 6. *A very general Fano hypersurfaces $X_d \subset \mathbb{P}^{n+1}$ is not ruled, in particular, not rational, if $d \geq 2\lceil \frac{n+3}{3} \rceil$.*

Kollar's approach introduced a powerful method by reduction to positive characteristic. The main reason of reduction to char$=p$ is to find a closed but non-exact differential i-form which is from a purely inseparable covering. The cohomology of $H^0(X, \Omega^i_X)$ is non-vanishing, which obstructs X to be rational. However, both inseparated covering and the desired form cannot exist in char$=0$.

Totaro's paper follows this philosophy, but strengthens the conclusion from irrationality to stable irrationality and strengthens into a larger range.

Definition 1. *A smooth projective variety X is stably rational if there exists an integer $m \geq 0$ such that*

$$X \times \mathbb{P}^m$$

is rational.

Rationality implies stable rationality. However, stable rationality has its own advantage. More precisely, stable rationality implies universal triviality of CH_0, which performs well under deformation. Universal triviality of CH_0 is equivalent to admitting a Chow-theoretic decomposition of the diagonal. We will discuss these later.

The main result discussed in this note is Totaro's theorem [Tot16].

Theorem 7. *Let $X_d \subset \mathbb{P}^{n+1}$ be a very general degree d hypersurface of dimension $n \geq 3$ over a characteristic 0 uncountable algebraic closed field. If*

$$d \geq 2\lceil \frac{n+2}{3} \rceil$$

then $\mathrm{CH}_0(X_d)$ is not universally trivial. In particular, X_d is not stably rational.

This result implies striking conclusions in special cases, including, very general quartic threefold and quartic fourfold is not rational. Moreover, it applies uniformly in all dimensions.

We summarize the proof here and discuss later in detail. By a specialization theorem, if a very general complex hypersurface had universally trivial CH_0, then a suitable resolution of a special fiber in positive characteristic will have universally trivial CH_0. Then, Totaro constructs a degeneration to characteristic 2 whose special fiber is an inseparable double covering. After resolving the singularities, one obtain nonzero global differential forms. But such differential form cannot exist for X_d with universally trivial CH_0.

Remark 2. *Totaro state his theorem over $\mathbb{C}$. While the specialization method held for arbitrary uncountable characteristic 0 algebraic closed field. Here, very general means that avoid a countably union closed subsets in the moduli of degree d hypersurfaces in $\mathbb{P}^{n+1}$.*

Remark 3. *It seems that Totaro's proof still held in $\mathrm{char} = 2$. To be more precisely, one make the DVR R, which is used for specialization argument, has $\mathrm{char} = 2$ fractional field rather than $\mathrm{char} = 0$. Then running the Totaro's argument without modification seems prove the following result:*

Let $X_d \subset \mathbb{P}^{n+1}$ be a very general degree d hypersurface of dimension $n \geq 3$ over a characteristic 0 uncountable algebraic closed field. If

$$d \geq 2\lceil \frac{n+2}{3} \rceil$$

then $\mathrm{CH}_0(X_d)$ is not universally trivial. In particular, X_d is not stably rational.

This modification seems to be not covered by [LS25] in quartics.

In this note I will focus on Totaro's proof. But for now let me list two other results to show that the method of decomposition of the diagonal is very actively used in the related problems.

Theorem 8. *(Voisin [Voi15]) The general quartic double solid with at most seven nodes does not admit a Chow-theoretic decomposition of the diagonal. In particular, it is not stably rational.*

Theorem 9. *(Colliot–Thelene–Piruka [CTP16]) A very general quartic threefold is not stably rational.*

In the end of this introduction, I state a generalization of Totaro due to [LS25] explain that Totaro's theorem is part of a larger modern program:

Theorem 10. *(Lange–Schreieder)*

- *Let k be a field of characteristic not equals 2. Let*
$$X \subset \mathbb{P}_k^{N+1}$$
be a very general hypersurface of degree $d \geq 4$ and dimension N. If
$$N \leq (d+1)2^{d-4},$$
then X does not admit a decomposition of the diagonal.
- *Let k be a field of characteristic 2. Let*
$$X \subset \mathbb{P}_k^{N+1}$$
be a very general hypersurface of degree $d \geq 5$ and dimension N. If
$$N \leq \frac{d+1}{3} 2^{d-4},$$
then X does not admit a decomposition of the diagonal.

In both cases, X is not stably rational.

Note that k is not required to be algebraically closed in this theorem.

2. Preliminary

2.1. **Chow Group of Zero-cycles.** Let X be a projective variety of dimension n over a field k.

Definition 2. *A zero-cycle on X is a finite formal sum*
$$z = \sum_i n_i [p_i],$$
where p_i are closed points on X and $n_i \in \mathbb{Z}$.

Let $Z_0(X)$ be the free abelian group of zero-cycles on X.

Note that every closed point $p \in X$ has finite residue field $k(p)/k$. One has a degree map
$$\deg : Z_0(X) \longrightarrow \mathbb{Z}, \ \sum_i n_i [p_i] \longmapsto \sum_i n_i [k(p_i) : k].$$

We introduce the following equivalent relation called rational equivalence to define the Chow group $\mathrm{CH}_0(X)$.

Let C be an algebraic curve over k and let $i : C \longrightarrow X$ be a proper morphism from C to X. We can properly push forward each divisor D on C into $\mathrm{CH}_0(X)$ by linearly extending the following map:

$$i_*([p]) = [k(p) : k(i(p))][i(p)].$$

Let $B_0(X)$ be the subgroup of $Z_0(X)$ generated by all such zero-cycles

$$i_*(\mathrm{div}_C(f)), \quad f \in k(C)^*,$$

where $\mathrm{div}_C(f))$ is the principle divisor associated to f and $i : C \longrightarrow X$ runs over all proper maps from all curves C to X.

Definition 3. *The Chow groups of zero-cycles* $\mathrm{CH}_0(X)$ *is defined to be the quotient group*

$$Z_0(X)/B_0(X).$$

Since X is projective, the degree map induces:

$$\deg : \mathrm{CH}_0(X) \longrightarrow \mathbb{Z}.$$

Example 1. *For the projective space* $\mathbb{P}^n_k$*,*

$$\mathrm{CH}_0(\mathbb{P}^n_k) = \mathbb{Z}.$$

In fact, any two closed points in $\mathbb{P}^n$ *lie on a line* $\mathbb{P}^1$ *and the degree map is surjective.*

The group $\mathrm{CH}_0(X)$ has good functoriality. Proper morphisms induce pushforwards on zero-cycles. Flat morphisms induce pullbacks. In particular, if F/k is a field extension, then base change gives a natural pullback

$$\mathrm{CH}_0(X) \longrightarrow \mathrm{CH}_0(X_F).$$

Definition 4. *Let X be a projective variety over a field k. We say that X is universally* CH_0*-trivial if for every field extension F/k, the degree map*

$$\deg : \mathrm{CH}_0(X_F) \longrightarrow \mathbb{Z}$$

is an isomorphism.

The condition universal is essential. It is much stronger than $\mathrm{CH}_0(X) = Z$ over the original ground field.

Definition 5. *A proper morphism $f : Y \longrightarrow X$ is called universally* CH_0*-trivial if for every field extension F/k, the pushforward*

$$f_* : \mathrm{CH}_0(Y_F) \longrightarrow \mathrm{CH}_0(X_F)$$

is an isomorphism.

Lemma 1.

$$\mathrm{CH}_0(X_F \times \mathbb{P}^m_F) \cong \mathrm{CH}_0(X_F).$$

Proof. Note that X_F is a proper retraction of $X_F \times \mathbb{P}^m_F$. So, $\mathrm{CH}_0(X_F)$ is a direct summand of $\mathrm{CH}_0(X_F \times \mathbb{P}^m_F)$. On the other hand, each point (x, y) in $\mathrm{CH}_0(X_F \times \mathbb{P}^m_F)$ can be moved rationally to (x, x_0) for a chosen base point in $\mathbb{P}^m_F$, see Example 1. □

Proposition 3. *If X and Y are smooth proper varieties over k and birational equivalent, then for every field extension F/k there is a natural isomorphism:*

$$\mathrm{CH}_0(X_F) \cong \mathrm{CH}_0(Y_F).$$

Proof. First suppose that $f : X \longrightarrow Y$ is a birational morphism. There is a dense open subset $V \subset Y$ such that f is an isomorphism over V. Let $U = f^{-1}(V)$. Since X_F and Y_F are smooth, the moving lemma for zero-cycles allows every zero-cycle to be moved away from the complements $X_F \setminus U_F$ and $Y_F \setminus V_F$. On the open sets $U_F \cong V_F$, the push-forward by f is an isomorphism. Thus $f_* : \mathrm{CH}_0(X_F) \longrightarrow \mathrm{CH}_0(Y_F)$ is bijective.

For a general birational map $X \dashrightarrow Y$, take the closure of its graph and resolve it. In character zero this gives a smooth proper variety W with birational morphisms

$$W \longrightarrow X, \quad W \longrightarrow Y.$$

And one can argue similarly. □

The proof uses the blow-up formula and Chow moving lemmas, see [Ful98].

Combining Lemma 1, we obtain:

Proposition 4. *Let X be a smooth projective variety over k. If X is stably rational, then X is universally* CH_0*-trivial.*

In application to singular special fibers, one usually has to check whether the resolution morphism is universally trivial or not. One method is to look at its fibers.

Theorem 11. *Let $f : X \longrightarrow Y$ be a proper morphism of varieties over k. Suppose that for every field extension F/k and every point $y \in Y_F$, the fiber X_y has universally trivial* CH_0 *over the residue field $F(y)$. Then f is universally* CH_0*-trivial.*

Proof. It is enough to prove that $f_* : \mathrm{CH}_0(X) \longrightarrow \mathrm{CH}_0(Y)$ is an isomorphism, since the same argument applies after replacing k by any field extension F/k.

Let $[y]$ be a closed point of Y, with residue field $L = k(y)$. By assumption, the fiber X_y is universally CH_0-trivial over L. In particular, the degree map gives $\mathrm{CH}_0(X_y) \cong \mathbb{Z}$. So there is a zero-cycle $\xi_y \in \mathrm{CH}_0(X_y)$ of degree one. Pushing ξ_y forward to X, we get a zero-cycle in X whose image in Y is $[y]$. This prove the surjectivity.

Let $\alpha \in Z_0(X)$ be a zero-cycle such that $f_*[\alpha] = 0$ in $\mathrm{CH}_0(Y)$. This means that $f_*\alpha$ is rationally equivalent to zero on Y. Hence, we may write

$$f_*\alpha = \sum_l (\nu_l)_* \operatorname{div}_{\widetilde{C}_l}(g_l),$$

where each $\nu_l : \widetilde{C}_l \longrightarrow Y$ is the normalization of an integral curve in Y, and $g_l \in k(\widetilde{C}_l)^*$.

Fix $\nu : \widetilde{C} \to Y$, $g \in k(\widetilde{C})^*$. Form the fiber product $X_{\widetilde{C}} := X \times_Y \widetilde{C}$ with projections $q : X_{\widetilde{C}} \to X$ and $p : X_{\widetilde{C}} \to \widetilde{C}$. Let η be the generic point of $\widetilde{C}$. By assumption, the generic fiber $X_\eta = X \times_Y \operatorname{Spec} k(\widetilde{C})$ is universally CH_0-trivial. In particular, the degree map $\deg : \mathrm{CH}_0(X_\eta) \longrightarrow \mathbb{Z}$ is an isomorphism, hence there exists a zero-cycle of degree 1 on X_η. Write such a cycle as

$$\xi_\eta = \sum_j m_j [P_j],$$

where the P_j are closed points of X_η.

Let $\Gamma_j \subset X_{\widetilde{C}}$ be the closure of P_j. Then $p_j = p|_{\Gamma_j} : \Gamma_j \to \widetilde{C}$ is generically finite of degree $d_j = [k(P_j) : k(\widetilde{C})]$. Since ξ_η has degree 1, we have $\sum_j m_j d_j = 1$. Pull back g to a rational function $p_j^* g \in k(\Gamma_j)^*$. Define a zero-cycle on X by

$$R_{\widetilde{C},g} = q_* \left(\sum_j m_j \ \mathrm{div}_{\Gamma_j}(p_j^* g) \right).$$

Since each $\mathrm{div}_{\Gamma_j}(p_j^* g)$ is a principal divisor on the curve Γ_j, we have $[R_{\widetilde{C},g}] = 0$ in $\mathrm{CH}_0(X)$.

On the other hand, pushing forward to Y, we get

$$f_* R_{\widetilde{C},g} = \nu_* \left(\sum_j m_j (p_j)_* \mathrm{div}_{\Gamma_j}(p_j^* g) \right).$$

For a generically finite morphism $p_j : \Gamma_j \to \widetilde{C}$ of degree d_j, one has $(p_j)_* \mathrm{div}_{\Gamma_j}(p_j^* g) = d_j \ \mathrm{div}_{\widetilde{C}}(g)$. Therefore,

$$f_* R_{\widetilde{C},g} = \nu_* \left(\sum_j m_j d_j \ \mathrm{div}_{\widetilde{C}}(g) \right) = \nu_* \mathrm{div}_{\widetilde{C}}(g).$$

Applying this construction to every pair $(\widetilde{C}_l, g_l)$, define

$$R := \sum_\ell R_{\widetilde{C}_\ell, g_\ell}.$$

Then $[R] = 0$ in $\mathrm{CH}_0(X)$ and $f_* R = f_* \alpha$ as actual zero-cycles on Y.

Let $\rho := \alpha - R$. Then $[\rho] = [\alpha] \in \mathrm{CH}_0(X)$. But now $f_* \rho = 0$ as an actual zero-cycle on Y (without modulo the equivalent relation).

We may divide ρ fiber by fiber. Since ρ has finite support, we can write

$$\rho = \sum_y \rho_y,$$

where each ρ_y is a zero-cycle supported on the fiber X_y over a closed point $y \in Y$. For each such y, $f_* \rho_y = \deg_{k(y)}(\rho_y)[y]$. Since $f_* \rho = 0$ as an actual zero-cycle on Y, and since $Z_0(Y)$ is freely generated by the closed points of Y, it follows that $\deg_{k(y)}(\rho_y) = 0$ for every y.

By the universal CH_0-triviality of X_y again, each degree-zero cycle ρ_y is zero in $\mathrm{CH}_0(X_y)$. Pushing forward by the closed immersion $i_y : X_y \hookrightarrow X$, we get $(i_y)_*[\rho_y] = 0 \in \mathrm{CH}_0(X)$. Therefore, $[\rho] = \sum_y (i_y)_*[\rho_y] = 0 \in \mathrm{CH}_0(X)$. Since $[\rho] = [\alpha]$, we conclude that $[\alpha] = 0 \in \mathrm{CH}_0(X)$. Thus $f_* : \mathrm{CH}_0(X) \to \mathrm{CH}_0(Y)$ is injective. □

2.2. **Decomposition of the Diagonal.** Let $\Delta_X \subset X \times X$ be the diagonal. It defines a class

$$[\Delta_X] \in \mathrm{CH}_n(X \times X).$$

The construction of the n-th Chow group CH_n is similar as CH_0 once one replaces points and curves by dimension n and $n+1$ subvarieties, respectively (for more details, see [Ful98]). A $\deg = 1$ zero-cycle x in X gives another n-cycle $X \times x$.

Definition 6. *Let X be a smooth projective variety of dimension n over k. We say that X admits a decomposition of the diagonal if there is a $\deg = 1$ zero-cycle x in X, a proper closed subset $D \subsetneqq X$, and a cycle $Z \in \mathrm{CH}_n(X \times X)$ supported on $D \times X$, such that*

$$[\Delta_X] = [X \times x] + Z \in \mathrm{CH}_n(X \times X).$$

Theorem 12. *Let X be a smooth projective variety over a field k. Then the following are equivalent:*

(i) *X is universally CH_0-trivial.*
(ii) *The class of the generic point η_X in $\mathrm{CH}_0(X_{k(X)})$ is equal to the base change of a $\deg = 1$ zero-cycle $z \in \mathrm{CH}_0(X)$*
(iii) *X admits an integral decomposition of the diagonal.*

Proof. (i) implies (ii). Choose $z \in \mathrm{CH}_0(X)$ a $\deg = 1$ cycle. Note that both the base change $z_{k(X)}$ and η_X are of $\deg = 1$ in $\mathrm{CH}_0(X_{k(X)})$. They must coincide since X is universally CH_0-trivial.

(ii) implies (iii). View $k(X)$ as the function field of the first factor of $X \times X$. Taking the generic point of the first factor identifies

$$\mathrm{CH}_0(X_{k(X)}) = \varinjlim_{U \subset X} \mathrm{CH}_n(U \times X),$$

where U runs over non-empty open subsets of X. Under this identification, the class $[\eta_X]$ is the restriction of the diagonal $[\Delta_X]$ to $\mathrm{Spec}\, k(X) \times X$, and $z_{k(X)}$ is the restriction of $[X \times z]$. Thus the assumption says that the class $[\Delta_X] - [X \times z]$ restricts to zero in $\mathrm{CH}_0(X_{k(X)})$. Therefore, there exists a non-empty open subset $U \subset X$ such that

$$([\Delta_X] - [X \times z])|_{U \times X} = 0 \in \mathrm{CH}_n(U \times X).$$

Let $D = X \setminus U$, we have the following exact sequence:

$$\mathrm{CH}_n(D \times X) \longrightarrow \mathrm{CH}_n(X \times X) \longrightarrow \mathrm{CH}_n(U \times X) \longrightarrow 0,$$

which shows that there is a cycle $Z \in \mathrm{CH}_n(D \times X)$ such that

$$[\Delta_X] - [X \times z] = Z \in \mathrm{CH}_n(X \times X).$$

This is precisely a decomposition of the diagonal. (iii) implies (i). Let F/k be any field extension. Base-changing the decomposition gives

$$[\Delta_{X_F}] = [X_F \times z_F] + Z_F \in \mathrm{CH}_n(X_F \times X_F),$$

with $\mathrm{Supp}\, Z_F \subset D_F \times X_F$. Let $\alpha \in \mathrm{CH}_0(X_F)$ have $\deg = 0$. By the moving lemma, we may represent α by a zero-cycle whose support is disjoint from D_F. Let a correspondence $\Gamma \in \mathrm{CH}_n(X_F \times X_F)$ act on zero-cycles by

$$\Gamma_*(\alpha) = p_{2*}(p_1^*\alpha \cdot \Gamma),$$

where p_1 and p_2 are the projections to the first and last factors, respectively. The diagonal acts as the identity, hence,

$$\alpha = (\Delta_{X_F})_*(\alpha).$$

On the other hand,

$$(X_F \times z_F)_*(\alpha) = \deg(\alpha) z_F = 0,$$

because $\deg(\alpha) = 0$. Moreover,

$$(Z_F)_*(\alpha) = 0,$$

because α is move away from D_F. Therefore, $\alpha = 0$. Combining $\deg(z_F) = 1$, we obtain:

$$\mathrm{CH}_0(X_F) \cong \mathbb{Z},$$

that is, X is universally CH_0-trivial. □

The equivalent conditions can help us prove the following vanishing result.

Proposition 5. *Let X be a smooth projective variety over a field k. If X is universally CH_0-trivial, then*

$$H^0(X, \Omega_X^i) = 0, \ \forall i > 0.$$

Proof. By Theorem 12, X admits a decomposition of the diagonal

$$[\Delta_X] = [X \times z] + Z \in \mathrm{CH}_n(X \times X),$$

where z is a $\deg = 1$ zero-cycle and Z is supported on $D \times X$ for some proper closed subset $D \subsetneqq X$. Let $U = X \setminus D$. Restricting the equality to $U \times X$, we obtain

$$[\Delta_X]|_{U \times X} = [U \times z] \in \mathrm{CH}_n(U \times X).$$

Let $\omega \in H^0(X, \Omega_X^i)$ with $i > 0$. A correspondence Γ acts on differential forms as following

$$\Gamma^*(\omega) = p_{1*}(p_2^*(\omega)|_\Gamma),$$

where p_{1*} is integral along fibers.

The diagonal correspondence acts as the identity. Therefore, after restricting to U, the above equality gives $\omega|_U = (U \times z)^*(\omega)$. But the correspondence $U \times z$ factors through the zero dimensional scheme z. This must give 0 since $\deg(\omega) = i > 0$. Hence $\omega|_U = 0$. But zero locus of ω must be closed. So, $\omega = 0$. □

We now prove the specialization argument used in the degeneration method.

Theorem 13. *Let R be a discrete valuation ring with fraction field K and residue field k. Let*

$$\mathcal{X} \longrightarrow \operatorname{Spec} R$$

be a flat projective family whose generic fiber is $X = \mathcal{X}_K$ of dimension n. Let $Y = \mathcal{X}_k$ be the special fiber. Suppose that X admits a resolution of singularities

$$\widetilde{X} \longrightarrow X$$

with $\widetilde{X}_{\overline{K}}$ is universally CH_0-trivial. Assume moreover that Y has a resolution of singularities

$$\tau : \widetilde{Y} \longrightarrow Y,$$

which is universally CH_0-trivial. Then $\widetilde{Y}$ is universally CH_0-trivial.

Proof. By assumption, there exists a resolve of singularities

$$\sigma : \widetilde{X} \longrightarrow X,$$

where $\widetilde{X}_{\overline{K}}$ is universally CH_0-trivial. After a finite extension of the DVR, we may assume that the decomposition of the diagonal on the geometric generic fiber is defined over K. Since the residue field is algebraically closed, this finite extension does not change the residue field.

Thus there exist a degree-one zero-cycle $\widetilde{z}$ on $\widetilde{X}$, a proper closed subset $\widetilde{D} \subsetneq \widetilde{X}$, and a cycle $\widetilde{Z}$ supported on $\widetilde{D} \times \widetilde{X}$ such that

$$[\Delta_{\widetilde{X}}] = [\widetilde{X} \times \widetilde{z}] + \widetilde{Z} \in \mathrm{CH}_n(\widetilde{X} \times \widetilde{X}).$$

We push this relation forward by $\sigma \times \sigma : \widetilde{X} \times \widetilde{X} \longrightarrow X \times X$.

Since σ is proper birational, we have

$$(\sigma \times \sigma)_*[\Delta_{\widetilde{X}}] = [\Delta_X].$$

Put $z := \sigma_* \widetilde{z}$. Then z is a degree-one zero-cycle on X, and

$$(\sigma \times \sigma)_*[\widetilde{X} \times \widetilde{z}] = [X \times z].$$

Therefore

$$[\Delta_X] = [X \times z] + Z \in \mathrm{CH}_n(X \times X),$$

where $Z := (\sigma \times \sigma)_* \widetilde{Z}$. Moreover Z is supported on $D \times X$, where $D := \sigma(\widetilde{D}) \subsetneqq X$. The subset D is proper because $\widetilde{D}$ has dimension strictly smaller than $\widetilde{X}$, and σ is birational.

Let $\overline{z}$, $\overline{D}$, and $\overline{Z}$ denote the closures of z, D, and Z inside the corresponding total spaces. Since

$$\mathcal{X} \longrightarrow \operatorname{Spec} R$$

is proper and flat, rational equivalences on the generic fiber specialize to rational equivalences on the special fiber.

Indeed, for an effective cycle $V \subset X$, define its specialization by taking the scheme-theoretic closure

$$\overline{V} \subset \mathcal{X}$$

and setting

$$\mathrm{sp}([V]) := [\overline{V}_k] \in Z_*(Y),$$

with the scheme-theoretic multiplicities in the special fiber. Here flatness guarantees that the special fiber of $\overline{V}$ has the expected dimension.

This construction respects rational equivalence. Suppose $\mathrm{div}_T(g)$ is a principal divisor on an integral subvariety $T \subset X$, with $g \in k(T)^*$. Let

$$\overline{T} \subset \mathcal{X}$$

be the closure of T. The rational function g gives a rational map $T \dashrightarrow \mathbb{P}^1_K$. Let

$$\Gamma_g \subset \overline{T} \times_R \mathbb{P}^1_R$$

be the closure of its graph. On the generic fiber, the difference between the fibers over 0 and ∞ is precisely $\mathrm{div}_T(g)$. After taking the special fiber, the cycle

$$(\Gamma_g)_k \subset \overline{T}_k \times_k \mathbb{P}^1_k$$

gives a rational equivalence on Y. Hence

$$\mathrm{sp}(\mathrm{div}_T(g)) = 0 \quad \text{in } \mathrm{CH}_*(Y).$$

Thus specialization of cycles descends to a well-defined homomorphism

$$\mathrm{sp} : \mathrm{CH}_*(X) \longrightarrow \mathrm{CH}_*(Y).$$

Applying this specialization map to the diagonal decomposition on X, we obtain

$$[\Delta_Y] = [Y \times z_0] + Z_0 \in \mathrm{CH}_n(Y \times Y),$$

where z_0 and Z_0 are the specializations of z and Z, respectively. The cycle z_0 is a degree-one zero-cycle on Y, and Z_0 is supported on $D_0 \times Y$, where $D_0 \subsetneq Y$ is the specialization of D. The subset D_0 is proper because the closure of D has relative dimension at most $n-1$ over the DVR.

Let $F := k(Y)$. Restricting the above decomposition to the generic point of the first factor gives

$$[\eta_Y] = [(z_0)_F] \in \mathrm{CH}_0(Y_F),$$

where $\eta_Y \in Y_F$ denotes the generic point of Y. The term Z_0 vanishes after this restriction, because its support is contained in $D_0 \times Y$, and the generic point of Y is not contained in D_0.

Now let

$$\tau : \widetilde{Y} \longrightarrow Y$$

be the resolution of singularities. Since τ is birational, we have $k(\widetilde{Y}) = k(Y) = F$. Let $\eta_{\widetilde{Y}} \in \widetilde{Y}_F$ be the generic point of $\widetilde{Y}$. Then

$$\tau_{F*}[\eta_{\widetilde{Y}}] = [\eta_Y] \in \mathrm{CH}_0(Y_F).$$

By assumption, τ is universally CH_0-trivial as a morphism. Hence the pushforward

$$\tau_* : \mathrm{CH}_0(\widetilde{Y}_E) \longrightarrow \mathrm{CH}_0(Y_E)$$

is an isomorphism for every field extension E/k.

Since z_0 has degree 1, there exists a unique class $\widetilde{z}_0 \in \mathrm{CH}_0(\widetilde{Y})$ such that

$$\tau_* \widetilde{z}_0 = z_0 \in \mathrm{CH}_0(Y).$$

Moreover $\widetilde{z}_0$ has degree 1, because proper pushforward preserves degrees of zero-cycles. After base change to F, we have

$$\tau_{F*}[(\widetilde{z}_0)_F] = [(z_0)_F] \in \mathrm{CH}_0(Y_F).$$

Combining the previous equalities gives

$$\tau_{F*}[\eta_{\widetilde{Y}}] = [\eta_Y] = [(z_0)_F] = \tau_{F*}[(\widetilde{z}_0)_F].$$

Since τ_{F*} is injective, it follows that

$$[\eta_{\widetilde{Y}}] = [(\widetilde{z}_0)_F] \in \mathrm{CH}_0(\widetilde{Y}_F).$$

This is the generic-point form of the decomposition of the diagonal for $\widetilde{Y}$: the class of the generic point of $\widetilde{Y}$ over $k(\widetilde{Y})$ is equal to the base change of a degree-one zero-cycle on $\widetilde{Y}$. By Theorem 12, $\widetilde{Y}$ is universally CH_0-trivial. □

2.3. **Inseparable Covering.** Let k be an algebraically closed field of char$= p > 0$. Let Z be a smooth variety of dimension n over k. Let L be a line bundle on X, and let

$$f \in H^0(X, L^p).$$

Locally, one may write f as a regular function. Let K be the function field of X, we assume $f \notin K^p$ on every local charts of X.

Let $\pi : E \longrightarrow X$ be the total space of L, then the line bundle π^*L on E has a tautological section y, which is defined as following:

$$y((z,e)) = e,$$

where $z \in X$ and $e \in E_z$.

Take a local chart U of X trivializing L such that $E|_U = \operatorname{Spec} \mathcal{O}_X(U)[u]$. Then the canonical section y is given by the vertical coordinate u. Now f pulls back to $(E, \pi^* L^p)$ and Y is defined by

$$y^p - \pi^* f = 0,$$

which is a well-defined closed sub-scheme in E. One can check this by trivializing $\pi^* L^p$ on E.

Restrict the projection π to $\pi : Y \longrightarrow X$, the fiber at closed point z is isomorphic to

$$\operatorname{Spec} k_z[y]/(y^p - f(z)).$$

So, $\pi : Y \longrightarrow X$ is a ramified deg $= p$ covering. We say such construction a cyclic covering.

Lemma 2. *The cyclic covering $\pi : Y \longrightarrow X$ is purely inseparable.*

Proof. This is a local problem. By passing to a open sub-scheme of X, we may assume L and L^p are both trivial line bundles. In this case, we may assume f is a function on X.

So, Y is defined by

$$\operatorname{Spec} \mathcal{O}_X[y]/(y^p - f).$$

Let $K(Y)$ be the function field of Y. Note that $f \notin K^p$ implies $y^p - f$ is irreducible. We obtain:

$$K(Y) = K[y]/(y^p - f),$$

the deg $= p$ purely inseparable extension. □

Now we consider the singular locus of Y. Locally, $Y \subset U \times \mathbb{A}^1$ is defined by $F = y^p - f = 0$. As an open subset of X, U is smooth. So, the singular locus of Y is defined by

$$\frac{\partial F}{\partial y} = \frac{\partial F}{\partial x_i} = 0. \quad i = 1, \ldots, n$$

where $x_1, \ldots, x_n$ is a locu étale coordinate of X.

Note that $\frac{\partial F}{\partial y} = p y^{p-1} = 0$ and $\frac{\partial F}{\partial x_i} = -\frac{\partial f}{\partial x_i}$. So, the singular locus of Y is exactly the critical locus of f.

Let Y^{sm} be the smooth locus of Y. Take differential of the equality $y^p - f = 0$ on Y. We know that

$$df = -d(y^p - f) = 0 \in \Omega^1_Y.$$

Lemma 3. *Let ω_X be the canonical line bundle on X. There is a natural nonzero homomorphism on Y^{sm}.*

$$\pi^*(\omega_X \otimes L^p)|_{Y^{sm}} \longrightarrow \Omega^{n-1}_{Y^{sm}}.$$

Locally, let e^p be the basis of L^p and $x_1, \ldots, x_n$ be the étale local coordinate on X, the local generator

$$(dx_1 \wedge \cdots \wedge dx_n) \otimes e^p$$

is sent, on the open set $D(f_i) := \{\frac{\partial f}{\partial x_i} \neq 0\}$, to

$$(-1)^{i-1} \frac{dx_1 \wedge \cdots \widehat{dx_i} \wedge \cdots \wedge dx_n}{f_i}.$$

Proof. We only need to check the construction above is agree on the overlap $D(f_i) \cap D(f_j)$ and is independent on the choice of x_i and e^p.

On the overlap $D(f_i) \cap D(f_j)$, wedge the relation

$$\sum_k f_k dx_k = 0$$

with

$$dx_1 \wedge \cdots \widehat{dx_i} \wedge \cdots \wedge \widehat{dx_j} \wedge \cdots \wedge dx_n,$$

we obtain:

$$(-1)^{i-1} f_i dx_1 \wedge \cdots \widehat{dx_j} \wedge \cdots \wedge dx_n = (-1)^{j-1} f_j dx_1 \wedge \cdots \widehat{dx_i} \wedge \cdots \wedge dx_n,$$

equivalently,

$$(-1)^{i-1} \frac{dx_1 \wedge \cdots \widehat{dx_i} \wedge \cdots \wedge dx_n}{f_i} = (-1)^{j-1} \frac{dx_1 \wedge \cdots \widehat{dx_j} \wedge \cdots \wedge dx_n}{f_j}.$$

Write $e_1 = ae$, then

$$f_1 = a^{-p} f, \quad f_{1,i} = a^{-p} f_i \text{ and } e'^p = a^p e^p.$$

Therefore, the independency of choice of e is a direct computation.

Finally, the construction can be read from the contraction of $\mathrm{grad}(f)$ into the volume form, which is coordinate independent. $\square$

Definition 7. *Let f be a local function on X. Assume f has a critical point at $z \in X$. Assume z is the origin point in a local coordinate $\{x_1, \ldots, x_n\}$ of X. Say f has admissible critical points at z if one of the following cases happens.*

- *$p \neq 2$ or if $p = 2$ and $\dim(X)$ is even, then the critical point is non-degenerated.*
- *$p = 2$ and $\dim(X)$ is odd, after a change of coordinates, one can write*

$$f = c + ax_1^2 + x_2x_3 + x_4x_5 + \cdots + x_{n-1}x_n + h_3,$$

with $\deg h_3 \geq 3$ and coefficient of x_1^3 nonzero.

Say f is admissible if all of its critical points are admissible.

For a section $f \in H^0(X, L^p)$, say f is admissible, if it is admissible as a local function on X.

Lemma 4. *For general $f \in H^0(X, \mathcal{O}_X(2a))$, $a \geq 2$, f has only admissible critical points.*

Proof. Fix a point $p \in X$. Since X is smooth, we may choose local coordinates $x_1, \ldots, x_n$ at p. Locally, after trivializing $\mathcal{O}_X(2a)$, the section f can be written as a function $f = c + \text{linear terms} + \text{quadratic terms} + \text{cubic terms} + \cdots$. The condition that p is a critical point is exactly that the linear terms vanish.

The admissibility condition is a condition on the next terms of this Taylor expansion. If n is even, one asks that the quadratic part be nondegenerate. Equivalently, after a change of local coordinates, the expansion begins as

$$f = c + x_1x_2 + x_3x_4 + \cdots + x_{n-1}x_n + \text{terms of degree at least 3}.$$

This is an open condition on the quadratic part.

If n is odd, then in characteristic 2 a quadratic form cannot be nondegenerate in odd dimension. In this case admissibility means that the quadratic part has the largest possible rank, namely $n-1$, and that the cubic term in the remaining direction is nonzero. Thus, after a change of coordinates, the expansion begins as

$$f = c + x_2x_3 + x_4x_5 + \cdots + x_{n-1}x_n + bx_1^3 + \text{terms of degree at least 3 without term } x_1^3,$$

with $b \neq 0$. Again, this is an open condition on the low-degree terms of f.

It remains to see that a general f satisfies this condition at every critical point. Since $a \geq 2$, sections of $\mathcal{O}_X(2a)$ have enough freedom to prescribe the Taylor expansion of f up to degree 3 at any chosen point of X. Indeed, locally on an affine chart, any polynomial in the local coordinates of degree at most 3 can be obtained as the beginning of the restriction to X of a homogeneous polynomial of degree $2a$.

For a fixed point p, the condition that p is a critical point imposes n independent linear conditions on f. Among those f for which p is critical, the condition that the critical point is not admissible imposes at least one additional independent closed condition on the quadratic and cubic terms. Hence the set of pairs (p, f) such that p is a non-admissible critical point of f is a proper codimension $\geq n+1$ closed subset of

$$X \times H^0(X, \mathcal{O}_X(2a)).$$

Its image in $H^0(X, \mathcal{O}_X(2a))$ is therefore a proper closed subset.

Consequently, a general section $f \in H^0(X, \mathcal{O}_X(2a))$ has no non-admissible critical points. Equivalently, all critical points of f are admissible. □

Theorem 14. *Suppose Y is constructed from an admissible section f.*

Let

$$\tau : \widetilde{Y} \longrightarrow Y$$

be the resolution obtained by blowing up the singular points. Assume $n = \dim(X) \geq 3$. Then the homomorphism constructed on Y^{sm} extends to a nonzero homomorphism:

$$\tau^*\pi^*(\omega_X \otimes L^p) \longrightarrow \Omega^{n-1}_{\widetilde{Y}}.$$

In particular, if

$$H^0(\widetilde{Y}, \tau^*\pi^*(\omega_X \otimes L^p)) \neq 0,$$

then

$$H^0(\widetilde{Y}, \Omega^{n-1}_{\widetilde{Y}}) \neq 0.$$

Proof. We begin by showing that the resulting rational $(n-1)$-form in Lemma 3 has no pole along the exceptional divisors of the resolution. Note that f has only isolated singular points since f is admissible. We may assume that the critical point $(x_1, \dots, x_n, y) = (0, \dots, 0, 0)$.

We first deal with the non-degenerated case. The local equation has the form

$$y^p = Q(x_1, \dots, x_n) + \text{terms of degree at least 3},$$

where Q is a non-degenerated quadratic form. If $p = 2$ and n is even, one may take

$$Q = x_1x_2 + x_3x_4 + \cdots + x_{n-1}x_n.$$

If $p \neq 2$, one may diagonalize Q. Blow up the origin and consider the chart where $x_1 = u$, and write $x_j = uz_j$, $j \geq 2$, $y = uw$.

In this chart, the strict transform is of the form:

$$u^{p-2}w^p = Q(1, z_2, \dots, z_n) + u \cdot h(u, z_2, \dots, z_n).$$

Because Q is non-degenerated, on every point of the exceptional divisor, one can choose some partial derivative of the right-hand side is a unit. Thus the strict transform is smooth.

Now f_i has leading term a nonzero multiple of u, the local generator of Kollár's form is

$$\eta_i = (-1)^{i-1} \frac{dx_1 \wedge \cdots \widehat{dx_i} \wedge \cdots \wedge dx_n}{f_i}.$$

Each differential dx_j with $j \neq 1$ has the form

$$dx_j = z_j du + u dz_j.$$

Hence the numerator is divisible by at least u^{n-2}. Since f_i is divisible by exactly one power of u, the quotient is divisible by u^{n-3}, and is therefore regular for $n \geq 3$. The same computation applies on the chart where y is the exceptional coordinate.

When $p = 2$ and n is odd. After replacing y by $y + cx_1$, the local equation may be written as

$$y^2 = x_2 x_3 + x_4 x_5 + \cdots + x_{n-1} x_n + b x_1^3 + \text{terms of order at least } 4, \;\; b \neq 0.$$

Again blow up the origin. On the chart $x_2 = u$, with $x_j = u z_j$, $y = uw$, the strict transform is

$$w^2 = z_3 + z_4 z_5 + \cdots + z_{n-1} z_n + b u z_1^3 + u^2 \cdot h,$$

which is smooth.

On the chart $x_1 = u$, the strict transform has the form

$$w^2 = z_2 z_3 + z_4 z_5 + \cdots + z_{n-1} z_n + bu + u^2 \cdot h.$$

The derivative with respect to u is a unit, so it is smooth along the exceptional divisor.

The same argument as the non-degenerated case implies the form is regular along the exceptional divisor.

Since the singularities are isolated and the above computation applies at every admissible critical point, the local homomorphisms glue to a global homomorphism

$$\tau^* \pi^* (\omega_X \otimes L^p) \longrightarrow \Omega^{n-1}_{\widetilde{Y}}.$$

It is nonzero because it is nonzero on Y^{sm} by Lemma 3.

Finally, $\widetilde{Y}$ is smooth and integral, so $\Omega^{n-1}_{\widetilde{Y}}$ is locally free and hence torsion-free. A nonzero homomorphism from a line bundle to a torsion-free sheaf has zero kernel. Therefore, it is injective. Taking global sections, a nonzero section of $\tau^* \pi^* (\omega_X \otimes L^p)$ gives a nonzero section of $\Omega^{n-1}_{\widetilde{Y}}$. Hence

$$H^0(\widetilde{Y}, \tau^* \pi^* (\omega_X \otimes L^p)) \neq 0,$$

implies

$$H^0(\widetilde{Y}, \Omega^{n-1}_{\widetilde{Y}}) \neq 0.$$

□

3. Proof of Totaro's Theorem

Now we are ready to prove the Totaro's theorem 7. We will to prove this theorem for coefficient field K, an uncountable char $= 0$ algebraically closed field.

The proof has two parts. First one deals with the even degree hypersurfaces. The method is to degenerate to an inseparable double covering in char $= 2$. Then one deals with odd degrees by degenerating to a reducible variety of the form $X_{2a} \cup H$. Then one uses a variant of the specialization criterion for reducible special fibers.

3.1. **Even Degree.** Let

$$d = 2a,$$

then the numerical hypothesis becomes

$$a \geq \lceil \frac{n+2}{3} \rceil.$$

Let R be a discrete valuation ring with uniformizer t. Let F be the fraction field of R and k be the residue field. Assume that F is of char $= 0$ and k is algebraically closed of char $= 2$.

For example, let k be an algebraically closed field k of char $= 2$, R can be the Witt vectors ring $W(k)$.

Consider the weighted projective space

$$S = \mathbb{P}_R(x_0, \ldots, x_{n+1}, y),$$

where $x_0, \ldots, x_{n=1}$ are of weight 1 and y is of weight a.

Define a closed subscheme

$$\mathcal{W} \subset S$$

by the equations

$$y^2 = f, \;\; g = ty.$$

On the generic fiber, one can invert t to eliminate y by writing $y = g/t$. Thus the generic fiber is isomorphic to the hypersurface X_{2a} defined by

$$g^2 = t^2 f$$

in $\mathbb{P}^{n+1}_F$. This is a hypersurface of degree $2a$. For a general choice of f and g, this generic fiber is smooth.

On the special fiber, where $t = 0$, the equations become

$$y^2 = f, \quad g = 0.$$

Let X be the deg $= a$ hypersurface in $\mathbb{P}^{n+1}_k$ defined by $g = 0$. Then special fiber is the double covering

$$\pi : Y \longrightarrow X,$$

given by $f \in \mathcal{O}_X(a)$.

This is a purely inseparable covering once $f \notin k(X)^2$, where $k(X)$ is the function field of X, by Lemma 2. We may further assume that f has only admissible critical points, see [Kol96]. The singular points of Y are the critical points of f.

Let

$$\tau : \widetilde{Y} \longrightarrow Y$$

be the blow-up of these singular points. By Theorem 14, this blow-up resolves Y.

Each exceptional divisor is a quadric of dim $= n-1$. In fact, take étale coordinates $\{x_1, \ldots, x_n\}$ of X. We may assume that the singularity is in the original point and the local equation of Y is given by

$$y^2 = x_1x_2 + \cdots + x_{n-1}x_n + \text{terms of degree at least 3},$$

if n is even;

$$y^2 = x_2x_3 + \cdots + x_{n-1}x_n + \text{terms of degree at least 3},$$

if n is odd.

Therefore, the exceptional divisor E is given by

$$y^2 = x_1x_2 + \cdots + x_{n-1}x_n,$$

if n is even;

$$y^2 = x_2x_3 + \cdots + x_{n-1}x_n,$$

if n is odd.

In the first case, it is a smooth quadratic.

In the second case, it is a degenerated quadratic in $\mathbb{P}^n_{[y:x_1:\cdots:x_n]}$. Consider an $n-2$ dimensional smooth quadratic Q given by $E \cap \{x_1 = 0\}$. Then each line from a point of Q to $v = [0:1:0:\cdots:0]$ lies in E. Therefore, E is a cone of a smooth quadratic with vertex v. So, each point is rationally equivalent to the cone vertex.

In both case, the exceptional divisors are universally CH_0-trivial. By Theorem 11, the resolution τ is universally CH_0-trivial.

Recall that Y is determined by section $f \in L = \mathcal{O}_X(a)$. So, we have

$$\omega_X \otimes L^2 \cong \mathcal{O}_X(a-n-2) \otimes \mathcal{O}_X(2a) \cong \mathcal{O}_X(3a-n-2).$$

The numerical hypothesis

$$a \geq \lceil \frac{n+2}{3} \rceil$$

means that

$$3a - n - 2 \geq 0.$$

Thus

$$H^0(\widetilde{Y}, \tau^*\pi^*(\omega_X \otimes L^2)) \neq 0.$$

By Theorem 14,

$$H^0(\widetilde{Y}, \Omega^{n-1}_{\widetilde{Y}}) \neq 0.$$

By Proposition 5, $\widetilde{Y}$ is not universally CH_0-trivial.

By Theorem 13, the generic fiber X_{2a} and $X_{2a,\overline{F}}$ of $\mathcal{W}$ is not universally CH_0-trivial. By Proposition 4, we know that $X_{2a,\overline{F}}$ cannot be stable rational.

3.2. **Odd Degree.** Now let

$$d = 2a+1, \quad a \geq \lceil \frac{n+2}{3} \rceil.$$

We construct as in Section 3.1. Let A be a discrete valuation ring with uniformizer π. Assume the residue field k of A is a char $= 2$ algebraically closed field and the fraction field K_1 of A is of char $= 0$.

Let K_0 be the algebraic closure of K_1

Consider the following

$$\mathcal{W} = \{y^2 = f, g = \pi y\} \subset S$$

family over $\operatorname{Spec} A$, where S is the weighted projective space, $\deg(f) = \deg(g) = a$ and f has only admissible singularities, see Lemma 4.

Let $X_1 \subset \mathbb{P}^{n+1}_{K_1}$ be a smooth hypersurface of $\deg = 2a$, which is the generic fiber of a family. Denote by F_{2a} the defining equation of X_1.

Let Y be the special fiber of this family, which is a purely inseparable covering of a $\deg = a$ hypersurface.

Let H_A be the following family

$$H_A = \{l = 0, g = \pi y\} \subset S.$$

And $H_1 = \{l = 0\} \subset \mathbb{P}^{n+1}_{K_1}$ be a hyperplane, which is isomorphic to the generic fiber of H_A. We assume H_1 intersects X_1 properly. Put

$$S_1 = X_1 \cap H_1.$$

Let H_0 be the special fiber of H_A. We also assume H_0 intersects Y properly. Put

$$Y_H = Y \cap H_0,$$

a hyperplane section of Y.

We mainly deal with $X_1 \times_{K_1} K_0$, $H_1 \times_{K_1} K_0$ and $S_1 \times_{K_1} K_0$. We denote them X, H and S, respectively.

Choose a general homogeneous polynomial G_{2a+1} of $\deg = 2a+1$. Let $R = K_0[[t]]$ with fraction field $F = K_0((t))$. Now define

$$\mathcal{W} = \{lF_{2a} + tG_{2a+1} = 0\} \subset \mathbb{P}^{n+1}_R.$$

Then $\mathcal{W} \longrightarrow \operatorname{Spec} R$ is a flat projective family. Its special fiber is

$$W_0 = \{lF_{2a} = 0\} = X \cup H$$

and its generic fiber is

$$W = \{lF_{2a} + tG_{2a+1} = 0\} \subset \mathbb{P}^{n+1}_F.$$

For general G_{2a+1}, this generic fiber is smooth.

We assume W is Fano, that is, $2a + 1 \le n + 1$, Otherwise $\omega_W \cong \mathcal{O}_W(2a + 1 - n - 2)$ has nontrivial global section. By Proposition 5 and 4, W cannot be universally CH_0-trivial and W cannot be stably rational.

Since the special fiber $V \coloneqq X \cup H$ is reducible, we need the following reducible specialization lemma, which is a variant of Theorem 13.

Lemma 5. *Let $\mathcal{W} \longrightarrow \operatorname{Spec} R$ be a flat proper family over a discrete valuation ring R with fraction field F and algebraically closed residue field K_0. Let W_0 be the special fiber and W be the generic fiber. Suppose that W admits a resolve of singularity*

$$\widetilde{W} \longrightarrow W$$

with $\widetilde{W}_{\overline{F}}$ universally CH_0-trivial. Then, for every extension field L of K_0, every $\deg = 0$ zero-cycle supported in the smooth locus $W^{sm}_{0,L}$ is zero in $\mathrm{CH}_0(W_{0,L})$.

Proof. We prove the statement first for $L = K_0$. The proof for an arbitrary extension field L/K_0 is obtained by the same argument after replacing the DVR R by a finite extension of R with residue field L.

Let

$$\alpha = \sum_i n_i[p_i]$$

be a degree-zero zero-cycle supported in $W_0^{\mathrm{sm}} \subset W_0$.

Since K_0 is algebraically closed, the points p_i are K_0-rational. We may replace R by its completion. This does not change the special fiber and does not affect the Chow group of the special fiber.

Because each p_i lies in the smooth locus of the special fiber, the morphism

$$\mathcal{W} \longrightarrow \operatorname{Spec} R$$

is smooth in a neighbourhood of p_i. By formal smoothness, after completing R, each point p_i lifts to a section

$$s_i : \operatorname{Spec} R \longrightarrow \mathcal{W}$$

whose special value is p_i. Let $\eta \coloneqq \operatorname{Spec} F$ be the generic point of $\operatorname{Spec} R$. The generic values $s_i(\eta)$ define a zero-cycle

$$\alpha_F \coloneqq \sum_i n_i[s_i(\eta)]$$

on the generic fiber $\mathcal{W}_F$. Since specialization preserves degrees and since $\deg(\alpha) = 0$, we have $\deg(\alpha_F) = 0$.

The sections s_i meet the special fiber in its smooth locus. Therefore, after shrinking around the images of the sections if necessary, their generic points lie in the smooth locus of the generic fiber. We choose the resolution

$$\widetilde{W} \longrightarrow W$$

to be an isomorphism over the smooth locus of W. Hence the zero-cycle α_F lifts uniquely to a degree-zero zero-cycle $\widetilde{\alpha}_F$ on $\widetilde{W}$.

By assumption, $\widetilde{W}_{\overline{F}}$ is universally CH_0-trivial. Therefore every degree-zero zero-cycle on $\widetilde{W}_{\overline{F}}$ is rationally equivalent to zero. In particular,

$$(\widetilde{\alpha}_F)_{\overline{F}} = 0 \in \mathrm{CH}_0(\widetilde{W}_{\overline{F}}).$$

The rational equivalence involved is defined over a finitely generated extension of F. After replacing R by a DVR dominating the original one and having the same residue field K, we may assume that this rational equivalence is defined over the fraction field. Thus

$$\widetilde{\alpha}_F = 0 \in \mathrm{CH}_0(\widetilde{W}).$$

Pushing forward along $\widetilde{W} \longrightarrow W$ gives

$$\alpha_F = 0 \in \mathrm{CH}_0(W).$$

Now specialize this rational equivalence to the special fiber. Concretely, write the rational equivalence on W as a finite sum of principal divisors

$$\alpha_F = \sum_j \operatorname{div}_{C_j}(g_j),$$

where the C_j are integral curves in the generic fiber and $g_j \in k(C_j)^*$. Taking the closures of the C_j inside $\mathcal{W}$, and taking the closures of the graphs of the rational functions g_j, shows that the specialization of the right-hand side is a rational equivalence on W_0. Since the specialization of α_F is precisely

$$\alpha = \sum_i n_i[p_i],$$

we obtain

$$\alpha = 0 \in \mathrm{CH}_0(W_0).$$

$\square$

Let $L = K_0(X)$ be the function field of X. Consider the base changes X_L, H_L, S_L and $V_L := (X \cup H)_L$. The generic point of X fives an L point $\eta_X \in X_L$. Choose an L point $h \in H_L \setminus S_L$. Both η_X and h lie in the smooth locus of V_L. Thus

$$\alpha = [\eta_X] - [h]$$

is a deg $= 0$ zero-cycle supported in the smooth locus of V_L.

Consider the natural exact sequence of the Chow group:

$$\mathrm{CH}_0(S_L) \longrightarrow \mathrm{CH}_0(X_L) \oplus \mathrm{CH}_0(H_L) \longrightarrow \mathrm{CH}_0(V_L) \longrightarrow 0.$$

Therefore, the pair $([\eta_X], -[h]) \in \mathrm{CH}_0(X_L) \oplus \mathrm{CH}_0(H_L)$ gives the deg $= 0$ cycle α

To show that $W_{\overline{F}}$ is not universally CH_0-trivial, it is remained to show $[\eta_X]$ is not in the image of the homomorphism

$$\mathrm{CH}_0(S_L) \longrightarrow \mathrm{CH}_0(X_L).$$

In fact, we then have that α is a nonzero deg $= 0$ zero-cycle in the smooth locus in $\mathrm{CH}_0(V_L)$. By Lemma 5, W is not universally CH_0-trivial.

Assume that $[\eta_X] \in \mathrm{CH}_0(X_L)$ is represented by a zero-cycle coming from S_L. This gives a supported decomposition of the diagonal:

$$[\Delta_X] = A + B \in \mathrm{CH}_n(X \times X),$$

where $\mathrm{Supp}(A) \subset X \times S$ and $\mathrm{Supp}(B) \subset D \times X$ for some proper closed subset $D \subsetneqq X$.

In fact, as in the proof of Theorem 12, restricting cycles on $X \times X$ to the generic point of the first factor gives

$$\mathrm{CH}_n(X \times X) \longrightarrow \mathrm{CH}_0(X_L).$$

Under this map, we have $[\Delta_X]$ maps to $[\eta_X]$.

By assumption, there exists a class $a_L \in \mathrm{CH}_0(S_L)$ whose pushforward to $\mathrm{CH}_0(X_L)$ is $[\eta_X]$.

Using the standard direct-limit description

$$\mathrm{CH}_0(S_L) = \varinjlim_{U \subset X} \mathrm{CH}_n(U \times S), \quad \mathrm{CH}_0(X_L) = \varinjlim_{U \subset X} \mathrm{CH}_n(U \times X),$$

where U runs over nonempty open subsets of the first copy of X, we may shrink U so that a_L is represented by a class $a_U \in \mathrm{CH}_n(U \times S)$ and the equality

$$(i_U)_* a_U = [\Delta_X]|_{U \times X}$$

holds in $\mathrm{CH}_n(U \times X)$.

Here $i_U : U \times S \longrightarrow U \times X$ is induced by $S \hookrightarrow X$.

By localization for the open immersion $U \times S \subset X \times S$, a_U extends to a class

$$\overline{a} \in \mathrm{CH}_n(X \times S).$$

Let

$$A = (\mathrm{id}_X \times i)_* \overline{a} \in \mathrm{CH}_n(X \times X),$$

where $i : S \hookrightarrow X$ is the inclusion.

Then $\mathrm{Supp}(A) \subset X \times S$ and $A|_{U\times X} = [\Delta_X]|_{U\times X}$.

Therefore $[\Delta_X] - A$ restricts to zero in $\mathrm{CH}_n(U \times X)$. Let $D := X \setminus U$. Since $U \neq \varnothing$, we have $D \subsetneq X$. By the localization exact sequence for $D \times X \subset X \times X$ with open complement $U \times X$, the class $[\Delta_X] - A$ comes from $CH_n(D \times X)$. Thus there exists a cycle class B supported on $D \times X$ such that

$$[\Delta_X] = A + B \in \mathrm{CH}_n(X \times X).$$

Now the deg $= 2a$ hypersurface X_1 degenerates to an inseparable double covering Y. This remains well-defined after we replace the K_1 by a finite extension K' of K_1 such that the support decomposition of $[\Delta_X]$ already held over K'.

Let $\tau : \widetilde{Y} \longrightarrow Y$ be the resolution obtained by blowing up the admissible critical points, and let $\tau|_{\widetilde{Z}} : \widetilde{Z} \longrightarrow Y_H$ be the corresponding resolution of the hyperplane section. The same as in Section 3.1, the resolutions τ and $\tau|_{\widetilde{Z}}$ are universally CH_0-trivial.

By specialization, the support decomposition of $[\Delta_X]$ gives the support decomposition of $[\Delta_Y]$. Sent the support decomposition of $[\Delta_Y]$ to $\mathrm{CH}_0(Y_{k(Y)})$, we know that $[\eta_Y]$ comes from a zero-cycle from Y_H, where $[\eta_Y]$ is the generic point of Y. By the universal CH_0-triviality, we know that $[\eta_{\widetilde{Y}}]$ comes from a zero-cycle from $\widetilde{Z}$, where $[\eta_{\widetilde{Y}}]$ is the generic point of $\widetilde{Y}$. This gives the following support decomposition of $[\Delta_{\widetilde{Y}}]$:

$$\Delta_{\widetilde{Y}} = A' + B' \in \mathrm{CH}_n(\widetilde{Y} \times \widetilde{Y}),$$

where $\mathrm{Supp}(A') \subset \widetilde{Y} \times \widetilde{Z}$ and $\mathrm{Supp}(B') \subset D' \times \widetilde{Y}$ for some proper closed subset $D' \subsetneq \widetilde{Y}$.

Now let the correspondence A' and B' act on $H^0(\widetilde{Y}, \Omega^i_{\widetilde{Y}})$. Since B' supports over a closed subset D' with strictly lower dimension, the action of B' is zero. If

$$\beta \in H^0(\widetilde{Y}, \Omega^i_{\widetilde{Y}})$$

restricts to zero on $\widetilde{Z}$, then A' also acts as zero on β, because the action of A' first pulls back to $\widetilde{Z}$. This implies $\beta = 0$. Hence the restriction map

$$H^0(\widetilde{Y}, \Omega^i_{\widetilde{Y}}) \longrightarrow H^i(\widetilde{Z}, \Omega^i_{\widetilde{Z}})$$

is injective for every $i > 0$.

We apply this with $i = n - 1$. Theorem 14 gives

$$H^0(\widetilde{Y}, \Omega^{n-1}_{\widetilde{Y}}) \neq 0$$

because $a \geq \lceil \frac{n+2}{3} \rceil$.

On the other hand, the hyperplane section Y_H has dimension $n-1$ and canonical bundle

$$\omega_{Y_H} \cong \mathcal{O}_Y(2a - n - 1),$$

where $2a - n - 1 \leq -1 < 0$ So we have

$$H^0(\widetilde{Z}; \Omega_{\widetilde{Z}^{n-1}}) = H^0(\widetilde{Z}, \omega_{\widetilde{Z}}) = H^0(Y_H, \omega_{Y_H}) = 0.$$

By the injectivity above forces

$$H^0(\widetilde{Y}, \Omega^{n-1}_{\widetilde{Y}}) = 0,$$

which is a contradiction.

In conclusion, $[\eta_X]$ is not in the image of $\mathrm{CH}_0(S_L)$.

3.3. **Comparing Coefficient Field.** There is a subtly that we only prove Totaro's theorem over $\overline{F}$, where F is the fraction field of a DVR, which is needed for the specialization argument.

Now we explain how to pass to the original coefficient field K.

Consider the parameter family of deg $= d$ hypersurfaces in $\mathbb{P}^{n+1}_{\mathbb{Z}}$

$$\mathcal{H}_{d,\mathbb{Z}} = \mathbb{P}H^0(\mathbb{P}^{n+1}_{\mathbb{Z}}, \mathcal{O}_{\mathbb{P}^{n+1}_{\mathbb{Z}}}(d)).$$

There is an open subset $\mathcal{H}^{sm}_{d,\mathbb{Z}}$ parametrizing all smooth deg $= d$ hypersurfaces in $\mathbb{P}^{n+1}_{\mathbb{Z}}$.

We explain that decomposition of the diagonal is a countable union of closed conditions in this family.

For a smooth degree d hypersurface X, to admit a decomposition of the diagonal means that there exist a zero-cycle z of degree 1, a proper closed subset $D \subsetneq X$, and a cycle Γ supported on $D \times X$, such that

$$[\Delta_X] = [X \times z] + \Gamma \in \mathrm{CH}_n(X \times X).$$

To make this into a closed condition in families, fix the numerical data of all cycles involved: Hilbert polynomials or Chow degrees for D, for z, for Γ, and for the auxiliary cycle on $X \times X \times \mathbb{P}^1$ which realizes the rational equivalence between the two sides. Once these numerical data are fixed, the possible tuples are parameterized by products of relative Hilbert schemes or Chow varieties over $\mathcal{H}^{\mathrm{sm}}_{d,\mathbb{Z}}$. These parameter spaces are projective over $\mathcal{H}^{\mathrm{sm}}_{d,\mathbb{Z}}$.

The condition that the fiber over $0 \in \mathbb{P}^1$ is the cycle Δ_X, and that the fiber over $\infty \in \mathbb{P}^1$ is $X \times z + \Gamma$, is closed. Therefore, for fixed numerical data, the image in $\mathcal{H}^{\mathrm{sm}}_{d,\mathbb{Z}}$ is a closed subset.

There are only countably many possible Hilbert polynomials and degrees. Hence the locus of smooth degree d hypersurfaces which admit a decomposition of the diagonal is contained in a countable union

$$\cup_{m=1}^{\infty} T_m$$

of closed subsets of $\mathcal{H}^{\mathrm{sm}}_{d,\mathbb{Z}}$.

For any algebraically closed field L, let $Z \subset \mathcal{H}^{\mathrm{sm}}_{d,L}$ be the sub-family of hypersurfaces X over L which admits decomposition of the diagonal. By the construction above, we have

$$Z \subset \cup_{m=1}^{\infty} T_{m,L}.$$

We only need to show that each T_m has strictly lower dimension. Assume there exists a T_m has full dimension, T_m must equal the whole family because T_m is closed. But our argument in Section 3.1 and 3.2 shows that one can always find an $\overline{F}$-point in $\mathcal{H}^{\mathrm{sm}}_{d,\overline{F}}$ which does not lie in $\cup_{m=1}^{\infty} T_{m,\overline{F}}$. A contradiction.

Then, we base change to the uncountable algebraically closed field K, the locus of stably ration hypersurfaces in $\mathcal{H}^{\mathrm{sm}}_{d,K}$ is contained in $\cup_{m=1}^{\infty}(T_m)_K$, a countable union of subvarieties of strictly lower dimension. This says exactly that a very general hypersurface of deg $= d$ in $\mathbb{P}^{n+1}_K$ is not stably rational.

4. Remaining parts

4.1. **Examples.** Totaro's proof of the stably irrational theorem is somehow constructive. More precisely, running the argument can help us construct hypersurfaces that are not stably rational. This is nontrivial. A counter-example is that one know that a very general cubic fourfold is not rational, while one can not write the equation of any irrational cubic fourfold explicitly. In this section, we consider the examples in [Tot16, Section 3].

Let $\mathbb{P}_{\mathbb{F}_{\not\succeq}}(1^{n+2}, 2)$ be the weighted projective space with the first $n+2$ homogeneous coordinate of degree 1 and last coordinate of degree 2. Suppose that over $\mathbb{F}_2$ the variety

$$Y = \{y^2 = f,\ g = 0\} \subset \mathbb{P}_{\mathbb{F}_2}(1^{n+2}, 2)$$

has only the admissible singularities used in the proof of the even degree case. Let $f_{\mathbb{Z}}$ and $g_{\mathbb{Z}}$ be integral lifts of f and g.

Then the hypersurface

$$X = \{g_{\mathbb{Z}}^2 - 4f_{\mathbb{Z}} = 0\} \subset \mathbb{P}^{n+1}$$

degenerates (along the DVR $W(\overline{\mathbb{F}}_2)$, special fiber $\overline{\mathbb{F}}_2$), modulo 2, to the characteristic 2 inseparable double cover Y. If X is smooth, the specialization theorem implies that $\mathrm{CH}_0(X_{\mathbb{C}})$ is not universally trivial ($\mathrm{Frac}(W(\overline{\mathbb{F}}_2)) \subset \mathbb{C}$). Hence $X_{\mathbb{C}}$ is not stably rational.

Example 2. *Consider the weighted projective space* $\mathbb{P}_{\mathbb{F}_2}(x_0, \ldots, x_5, y)$*, where* $\deg x_i = 1, \deg y = 2$*. Let*

$$g = x_0x_1 + x_2x_3 + x_4x_5$$

and

$$f = x_0^3x_2 + x_0x_3^3 + x_1^3x_4 + x_1^2x_2x_4 + x_1x_4^3 + x_2^3x_5 + x_3x_5^3.$$

Then the 4*-fold*

$$Y = \{y^2 = f,\ g = 0\} \subset \mathbb{P}_{\mathbb{F}_2}(x_0, \ldots, x_5, y)$$

has only the admissible singularities.

Taking the same polynomials as integral lifts, define a quartic hypersurface

$$X = \{g_{\mathbb{Z}}^2 - 4f_{\mathbb{Z}} = 0\} \subset \mathbb{P}^5.$$

Explicitly,

$$\begin{aligned} 0 = {} & x_0^2x_1^2 - 4x_0^3x_2 + 2x_0x_1x_2x_3 + x_2^2x_3^2 - 4x_0x_3^3 - 4x_1^3x_4 \\ & - 4x_1^2x_2x_4 - 4x_1x_4^3 - 4x_2^3x_5 + 2x_0x_1x_4x_5 + 2x_2x_3x_4x_5 \\ & + x_4^2x_5^2 - 4x_3x_5^3. \end{aligned}$$

This quartic 4*-fold is smooth. Therefore* $\mathrm{CH}_0(X_{\mathbb{C}})$ *is not universally trivial, and hence* $X_{\mathbb{C}}$ *is not stably rational.*

Example 3. *Consider the weighted projective space* $\mathbb{P}_{\mathbb{F}_2}(x_0, \ldots, x_4, y)$*, where* $\deg x_i = 1, \deg y = 2$*.*

Let

$$g = x_0x_1 + x_2x_3 + x_4^2$$

and

$$f = x_0^3x_2 + x_1^3x_2 + x_0x_1^2x_4 + x_0x_2^2x_4 + x_3^3x_4.$$

Then the 3*-fold*

$$Y = \{y^2 = f,\ g = 0\}$$

has only the admissible singularities.

For the integral lift, take

$$g_{\mathbb{Z}} = x_0x_1 + x_2x_3 + x_4^2$$

and

$$f_{\mathbb{Z}} = x_0^3x_2 + x_1^3x_2 + 2x_2^4 + x_0x_1^2x_4 + x_0x_2^2x_4 + x_3^3x_4.$$

The extra term $2x_2^4$ vanishes modulo 2, so this is still a lift of the same polynomial over $\mathbb{F}_2$.

Define a quartic hypersurface

$$X = \{g_{\mathbb{Z}}^2 - 4f_{\mathbb{Z}} = 0\} \subset \mathbb{P}^4.$$

Explicitly,

$$\begin{aligned} 0 = {} & x_0^2x_1^2 - 4x_0^3x_2 - 4x_1^3x_2 - 8x_2^4 + 2x_0x_1x_2x_3 + x_2^2x_3^2 \\ & - 4x_0x_1^2x_4 - 4x_0x_2^2x_4 - 4x_3^3x_4 + 2x_0x_1x_4^2 + 2x_2x_3x_4^2 + x_4^4. \end{aligned}$$

This quartic 3-fold is smooth. Therefore $\mathrm{CH}_0(X_{\mathbb{C}})$ is not universally trivial, and hence $X_{\mathbb{C}}$ is not stably rational.

4.2. **Rationality in families.** We finish by discussing Totaro's application to rationality in families. The question is the following: suppose

$$\mathcal{X} \longrightarrow B$$

is a flat family of projective varieties, and suppose that the general fiber is rational. Must the special fiber be rational?

smooth projective varieties this question is subtle and remains open in many cases. However, if one allows singular special fibers, rationality need not specialize. A classical example is a degeneration of smooth cubic surfaces to the projective cone over a smooth plane cubic curve. The smooth cubic surfaces are rational, but the cone over an elliptic curve is not rational. In that example, the special fiber has log canonical singularities, but it is not klt.

Totaro proves that rationality can fail to specialize even when the special fiber has klt singularities.

We first recall the terminology. Let X be a normal variety such that K_X is $\mathbb{Q}$-Cartier. Let

$$\mu : Y \longrightarrow X$$

be a log resolution (exceptional divisors are normal crossing). Then one can write

$$K_Y = \mu^* K_X + \sum_i a_i E_i,$$

where the E_i are exceptional divisors and the numbers $a_i \in \mathbb{Q}$ are called discrepancies.

We say that X has log canonical singularities if $a_i \geq -1$ for all i.

We say that X has Kawamata log terminal singularities, or klt singularities, if $a_i > -1$ for all i.

Thus klt singularities are milder than log canonical singularities.

The example of the cone over a smooth cubic curve is log canonical but not klt. Indeed, the elliptic curve has trivial canonical bundle, and the discrepancy of the exceptional divisor over the vertex is exactly -1. Totaro's construction improves this example by producing special fibers whose singularities are klt.

Corollary 1. *For every integer $m \geq 4$, there exists a flat family*

$$\mathcal{X} \longrightarrow \mathbb{A}^1_{\mathbb{C}}$$

of projective m-folds such that every fiber over $\mathbb{A}^1_{\mathbb{C}} \setminus \{0\}$ is smooth and rational, while the fiber over 0 is klt and not rational.

Proof. Choose an integer d such that

$$2\left\lceil \frac{m+1}{3} \right\rceil \leq d \leq m.$$

Such a d exists for every $m \geq 4$. By Totaro's theorem, there is a smooth degree d hypersurface

$$W \subset \mathbb{P}^m_{\mathbb{C}}$$

which is not stably rational. We know that W is Fano.

Now embed $\mathbb{P}^m$ by the d-uple Veronese embedding

$$\nu_d : \mathbb{P}^m \hookrightarrow \mathbb{P}^N.$$

Let $X := \nu_d(\mathbb{P}^m) \subset \mathbb{P}^N$. Then X is isomorphic to $\mathbb{P}^m$. Moreover, hyperplane sections of $X \subset \mathbb{P}^N$ correspond exactly to degree d hypersurfaces in $\mathbb{P}^m$. Therefore we may choose a hyperplane $H_0 \subset \mathbb{P}^N$ such that $X \cap H_0 \simeq W$.

Let $C(X) \subset \mathbb{P}^{N+1}$ be the projective cone over X, with vertex v. Choose a pencil of hyperplanes $\{H_t\}_{t\in\mathbb{A}^1}$ in $\mathbb{P}^{N+1}$ such that H_t does not contain v for $t \neq 0$, while H_0 contains v, and the induced hyperplane on $\mathbb{P}^N$ cuts X along W.

Define

$$\mathcal{X}_t := C(X) \cap H_t.$$

Then

$$\mathcal{X} \longrightarrow \mathbb{A}^1$$

is a flat family of projective m-folds. For $t \neq 0$, the hyperplane H_t does not contain the vertex of the cone. Projection from the vertex identifies

$$C(X) \cap H_t \simeq X \simeq \mathbb{P}^m.$$

Thus every nonzero fiber is smooth and rational.

On the other hand, the special fiber is

$$\mathcal{X}_0 = C(W),$$

the projective cone over W. Since a cone over W is birational to $W \times \mathbb{P}^1$, if $C(W)$ are rational, then $W \times \mathbb{P}^1$ would be rational. This would imply that W is stably rational, contradicting the choice of W. Therefore $\mathcal{X}_0$ is not rational.

It remains to check that the special fiber has klt singularities. The only singular point of $C(W)$ is the vertex. Let $L := \mathcal{O}_W(d)$ be the polarization induced by the Veronese embedding. The blow-up

$$\mu : \widetilde{C}(W) \longrightarrow C(W)$$

of the vertex of the cone has exceptional divisor $E \simeq W$. The discrepancy of this exceptional divisor is controlled by the relation between L and $-K_W$.

In fact, the exceptional locus consists only of E, we can write

$$K_{\widetilde{C}(W)} = \mu^* K_C + aE$$

for some rational number a. This number a is the discrepancy of the exceptional divisor E. Restricting this equality to E, the term $\mu^* K_{C(W)}|_E$ is trivial because E is contracted to the vertex. Hence $K_{\widetilde{C}(W)}|_E = aE|_E$. On the other hand, by adjunction,

$$(K_{\widetilde{C}(W)} + E)|_E = K_E = K_W.$$

Thus

$$aL = -aE|_E = -K_{\widetilde{C}(W)}|_E = -K_W + E|_E = -K_W - L.$$

Since $-K_W \cong \mathcal{O}_W(m+1-d)$, we have

$$-K_W \cong \frac{m+1-d}{d} L$$

as a $\mathbb{Q}$-line bundle. Because $d \leq m$, the coefficient $\frac{m+1-d}{d}$ is positive. Therefore

$$\mathcal{X}_0 = C(W)$$

has klt singularities. □

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